On Renormalization Group Flows and Polymer Algebras

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Abstract

In this talk methods for a rigorous control of the renormalization group (RG) flow of field theories are discussed. The RG equations involve the flow of an infinite number of local partition functions. By the method of exact beta-function the RG equations are reduced to flow equations of a finite number of coupling constants. Generating functions of Greens functions are expressed by polymer activities. Polymer activities are useful for solving the large volume and large field problem in field theory. The RG flow of the polymer activities is studied by the introduction of polymer algebras. The definition of products and recursive functions replaces cluster expansion techniques. Norms of these products and recursive functions are basic tools and simplify a RG analysis for field theories. The methods will be discussed at examples of the Φ^4-model, the $O(N)$ σ-model and hierarchical scalar field theory (infrared fixed points).

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1 Introduction

There are several goals in constructive field theory. The first one is a rigorous definition of Euclidean functional-integrals. There exists many examples for a construction of special functional-integrals (see e. g. [GJ73, MS77, MP85, GJ87, FMRS87, GK83, GK85, B88a, B88b, BY90, DH91, R91, DH92, DH93]). But a general definition of the Euclidean functional-integral is still lacking. The second goal in constructive field theory is to compute functional-integrals by approximation methods. The functional-integrals can be represented by convergent series expansions in terms of finite-dimensional integrals (cp. [MP89]). This is in analogy to conventional perturbation theory where functional-integrals are expressed in terms of (finite-dimensional) Feynman-integrals. The definition and construction of functional-integrals lead to a proof of the existence of ultraviolet- resp. infrared fixed points. The problem is the infinite (not denumerable) number of degrees of freedom which is connected to ultraviolet and infrared problems. The large field problem is connected to the divergence of standard power series expansions. A third aim is to represent the construction in such a way that all intermediary steps can be done in a finite number of well-defined computations.

There exists several tools to perform such a program. A field theory which is represented by a functional-integral can be studied by means of Wilson's renormalization group (RG) [W71, WK74, W83]. Thereby, the original functional-integral is represented by a RG flow of effective functional-integrals. These effective functional-integrals are simpler to define than the original functional-integral. The RG flow of the effective functional-integrals has to be controlled. The effective functional-integrals can be further analyzed by methods used in statistical mechanics, especially by the introduction of polymer systems (cp. [GrK71, MP85]). A suitable defined polymer system can control large field contributions and solves the large volume problem. Thereby, the effective systems represented by effective functional-integrals will be decomposed into finite subsystems. The effective systems depend on an infinite number of degrees of freedom. Their control is reduced to the problem of analyzing the RG flow of finite subsystems. In a RG analysis, using conventional perturbation theory, one distinguishes between relevant and irrelevant parts. Likewise, there are relevant and irrelevant parts of the effective finite subsystems. The relevant part depends only on a finite number of parameters. The flow of the irrelevant part can be controlled by an application of fixed point theorems. Thereby, the control of the RG flow of effective systems is reduced to a RG flow defined in a finite-dimensional parameter space. This method is called the *method of exact beta-functions*. For a control of effective systems one introduces norms for polymer activities. A suitable definition of norms and polymer systems is the technical core in the construction of field theoretic models.

It is the aim of this talk to review old and provide new tools and definitions for such a program. A test for simplicity of these methods is the implementabil-

ity on a computer.

This paper is organized as follows. We start in Sect. 2 with the introduction to general renormalization group transformations (RGT). Then, we consider the special example of the Kadanoff-Wilson (linear) block spin transformation. This RGT were firstly applied to field theory by Gawędzki and Kupiainen (cp. [GK80, GK83, GK85]). Next, we consider the definition of the nonlinear block spin RGT at the example of the nonlinear $O(N)$ σ-model.

Section 3 introduces a general polymer systems and presents the definition of polymer activities by introducing an exponential function EXP. The exponential function EXP is defined by a product $\circ$. This $\circ$-product was also used by Brydges and Yau, Dimock and Hurd [BY90, DH91] and has its origin in a product defined for problems in statistical mechanics by Ruelle [Ru69]. Ruelle's product differs from the $\circ$-product used in constructive field theory by an important point[2]. It does not allow an overlap of indices, whereas the $\circ$-product does not allow an overlap of lattice points. This property of nonoverlapping of lattice points is essential for a control of large fields. The RG flow of the effective systems is represented by the RG flow of polymer activities. The effective subsystems are defined on lattices.

One RG step can be decomposed into four steps. The first step is called integration step. In this step high momentum fields are integrated out. This can be done recursively, using a $\times_\Gamma$-product. After this step the correlation length becomes larger. Then, the activities are defined on a coarser lattice (coarsening step). The coarsening step can be performed in a recursive way by introducing a mapping E_A. The polymer acivities are defined in such a way that the localization property holds. A polymer activity $A(P|\psi)$ for a polymer P and field ψ obeys the localization property if $A(P|\psi)$ depends only on $\psi(y)$ for $y \in P$. This localization property of the polymer activities makes a third step (localization step) necessary. The localization step can be performed recursively, like the integration step, by using a $\times_{\mathcal{A}}$-product. The fourth and last step rescales the fields such that the new effective polymer activities lives on the same lattice as the polymer activities before the RG step.

Section 4 presents the general method of exact beta-functions. As examples we consider the RG flow of the Φ^4-model and hierarchical RG fixed points. Koch and Wittwer [KW86, KW91] applied the method of exact beta-functions to construct the double-well fixed point in 3 dimensions. These method is a candidate for the construction of field theories with no small coupling constants.

The split into relevant and irrelevant parts and the method of renormalization and repolymerization for the flow of effective polymer systems are studied in Sect. 5. For repolymerization a further RG step is necessary (repolymerization step).

Section 6 presents a norm for the polymer activities and shows how this norm behaves under RG steps. It will be shown that large fields are controlled

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[2]The author thanks D. Brydges for this comment

by the method of exponential pinning.

2 Renormalization Group Transformations

Our main object of interest is the generating functional of Euclidean Greens functions (partition function). It is the following infinite-dimensional integral

$$Z = \int [D\phi]\, \mathcal{Z}(\phi), \qquad [D\phi] = \prod_{z \in \mathbf{R}^d} d\phi(z) \ ,$$

where $\mathcal{Z}(\phi)$ is a real-valued function, called *Boltzmannian*. We restrict our attention here to real-valued scalar fields ϕ. In renormalization group (RG) investigations the computation of Z is performed stepwise. Let us consider the definition of one RG step. Define new fields Φ and a function $P(\Phi, \phi)$ which obeys

$$\int [D\Phi]\, P(\Phi, \phi) = 1 \ . \tag{1}$$

Then, the *renormalization group transformation* is defined as follows

$$\mathcal{Z}'(\Phi) = \int [D\phi]\, P(\Phi, \phi)\mathcal{Z}(\phi) \ . \tag{2}$$

Eqs. (1) and (2) imply

$$Z = \int [D\phi]\, \mathcal{Z}(\phi) = \int [D\Phi]\, \mathcal{Z}'(\Phi) \ .$$

Thus the new Boltzmannian $\mathcal{Z}'$ and the new field Φ can be used to compute the partition function Z. This RG procedure can be repeated and the result is the *RG flow of effective Boltzmannians* :

$$\mathcal{Z} \longrightarrow \mathcal{Z}' \longrightarrow \mathcal{Z}'' \longrightarrow \dots \ .$$

The definition of the RG is chosen in such a way that the effective Boltzmannians depend on fewer and fewer degrees of freedom. Thus, the task of computing Z by an infinite-dimensional integral is solved by computing an infinite number of RG steps. Instead of considering the RG flow of effective Boltzmannians, it is better to consider the RG flow of effective polymer activities. In this way the RG transformations can be represented by finite-dimensional integrals.

Before coming to the definition of polymer activities, we will study two examples of RG transformations. The first example is the *Kadanoff-Wilson (linear) block spin transformation*. This method was applied to field theory by Gawędzki and Kupiainen [GK80, GK83, GK85]. The second example is the nonlinear block spin transformation applied to the nonlinear σ-model. For a definition of a RG transformation introduce new fields Φ and the function

$P(\Phi, \phi)$ which obeys eq. (1). Let us suppose that the original field ϕ lives on the lattice $\Lambda := (a\mathbf{Z})^d$ and define the integral for the partition function Z by $[D\phi] := \prod_{y \in \Lambda} d\phi(y)$. Decompose the lattice Λ into hypercubes of side length La, where L is a fixed number, $L \in \{2, 3, \ldots\}$. The center points of these hypercubes (blocks) build also a lattice with side length La. This lattice is called *block lattice* $\Lambda' := (La\mathbf{Z})^d$. A site y of the lattice Λ is contained in a site x of the block lattice Λ' if y is contained in the block with center point x. In this case we write $y \underline{\in} x$. For a field $\phi : \Lambda \to \mathbf{R}$, define the *block spin field* : $C\phi : \Lambda' \to \mathbf{R}$ by

$$C\phi(x) := \beta \sum_{y:\, y \underline{\in} x} \phi(y), \qquad x \in \Lambda' \ .$$

β is a positive real number, called *scaling parameter*. Finally, the special RG transformation, called *block spin transformation*, is defined by

$$\mathcal{Z}'(\Phi) = \int [D\phi]\, P_{\alpha,\beta}(\Phi, \phi) \mathcal{Z}(\phi) \ ,$$

where $P_{\alpha,\beta}$ is defined by

$$P_{\alpha,\beta}(\Phi, \phi) := \mathcal{N}_{\alpha,\beta} \, \exp\{-\frac{\alpha}{2} \sum_{x \in \Lambda'} (\Phi(x) - (C\phi)(x))^2\} \ .$$

α is a positive real parameter and $\mathcal{N}_{\alpha,\beta}$ is a normalization constant such that eq. (1) holds.

We represent the Boltzmannian by a free propagator u and interaction V

$$\mathcal{Z}(\phi) = \exp\{-\frac{1}{2}(\phi, u^{-1}\phi) - V(\phi)\} \ .$$

$(\cdot, \cdot)$ is the canonical bilinear form. Then, the RG transformation reads

$$\mathcal{Z}'(\Phi) = \mathcal{N}_{\alpha,\beta} \, \exp\{-\frac{1}{2}(\Phi, {u'}^{-1}\Phi)\} \, \exp\{-V'(\Phi)\} \ ,$$

where the *effective interaction* V' is defined by

$$\exp\{-V'(\Phi)\} := \int [D\zeta] \, \exp\{-\frac{1}{2}(\zeta, \Gamma^{-1}\zeta)\} \exp\{-V(\zeta + \mathcal{A}\Phi)\} \ .$$

The *fluctuation propagator* Γ and the *block spin propagator* u' are

$$\Gamma := (u^{-1} + \alpha C^T C)^{-1}, \qquad u' := (\alpha - \alpha^2 C\Gamma C^T)^{-1} \ .$$

The $\mathcal{A}$-operator, which maps a field defined on Λ' to fields defined on Λ, is given by $\mathcal{A} := uC^T {u'}^{-1}$. The normalized Gaussian measure with mean zero is defined by

$$d\mu_\Gamma(\zeta) := \mathcal{N} \, [D\zeta] \, \exp\{-\frac{1}{2}(\zeta, \Gamma^{-1}\zeta)\} \ .$$

The RG transformation, for the effective interaction V, reads

$$e^{-V'(\Psi)} = \int d\mu_\Gamma(\phi) e^{-V(\phi + \mathcal{A}\Psi)} \ .$$

Let the free propagator u be a Gaussian fixed point. Then self-similarity holds

$$u'(Ly, Ly') = L^{2-d(+\epsilon)} u(y, y'), \qquad y, y' \in \Lambda \ .$$

Replacing the field $\Phi(\cdot)$ by the rescaled one, $L^{1-\frac{d}{2}(+\frac{\epsilon}{2})}\Phi(\frac{\cdot}{L})$, we obtain the RG transformation after rescaling

$$e^{-V'(\Phi)} = \int d\mu_\Gamma(\zeta) e^{-V(\zeta + L^{1-\frac{d}{2}(+\frac{\epsilon}{2})}\mathcal{A}(\Phi(\frac{\cdot}{L})))} \ .$$

The field Φ lives on the original lattice Λ but with reduced correlation length. This procedure can be iterated and the result is the following RG flow

$$e^{-V} \longrightarrow e^{-V'} \longrightarrow e^{-V''} \longrightarrow \cdots \ .$$

Best localization properties are obtained for $\alpha = O(1)$ (see Bell and Wilson [BW74]).

We consider a second example for a RG transformation. This is the nonlinear block spin transformation at the example of the nonlinear $O(N)$ σ-model. Define a measure on the $N-1$-dimensional unit sphere S^{N-1}

$$\int_{S^{N-1}} [d\sigma] := \int \prod_{z \in \Lambda} d\sigma_1(z) \cdots d\sigma_N(z) \delta(\sigma^2(z) - 1)$$

and a partition function

$$Z = \int_{S^{N-1}} [d\sigma] e^{-V(\sigma)} \ ,$$

where the interaction V is defined by nearest-neighbor couplings

$$V(\sigma) := \beta \sum_{\mu=1}^{d} \sum_{y \in \Lambda} \left(1 - \sigma(y) \cdot \sigma(y + \hat{\mu})\right) \ .$$

$\hat{\mu}$ is a vector in μ-direction with length of one lattice spacing a. Let $\mu : \Lambda' \to S^{N-1}$ be a unit vector field on the block lattice Λ'. Then, the nonlinear block spin RG transformation is defined by

$$\mathcal{Z}'(\mu) := \int_{S^{N-1}} [d\sigma] P(\mu, \sigma) e^{-V(\sigma)} \ ,$$

where

$$P(\mu, \sigma) := \exp\{-W_\kappa(\sigma)\} \prod_{x \in \Lambda'} \exp\{\beta\kappa\mu(x) \cdot \sum_{y \subseteq x} \sigma(y)\} \ .$$

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W_κ is defined such that

$$\int_{S^{N-1}} [d\mu] P(\mu, \sigma) = 1$$

holds. A simple computation shows

$$
\begin{aligned}
W_\kappa(\sigma) &= \sum_{x \in \Lambda} \ln \left(\int_{S^{N-1}} [d\mu] \, \exp\{\kappa\beta\mu \cdot \sum_{y \underline{\subseteq} x} \sigma(y)\} \right) \\
&= const + \sum_{x \in \Lambda} \ln \left(\frac{I_{\frac{N-2}{2}}(\kappa\beta| \sum_{y \underline{\subseteq} x} \sigma(y)|)}{(\kappa\beta| \sum_{y \underline{\subseteq} x} \sigma(y)|)^{\frac{N-2}{2}}} \right) \quad .
\end{aligned}
$$

$I_\nu(z)$ is the Bessel function represented by the series expansion

$$I_\nu(z) = \frac{z^\nu}{2^\nu} \sum_{k=0}^\infty (-1)^k \frac{z^{2k}}{2^{2k} k! \Gamma(\nu + k + 1)} \quad .$$

We have seen how to compute partition functions iteratively by the introduction of RG transformations. RG transformations are given by infinite-dimensional integrals. For a computation of RG steps one has to reduce these infinite-dimensional integrals to finite-dimensional ones. This problem corresponds to the infinite volume problem in statistical mechanics and can be solved by cluster expansion methods (cp. [B84]) or equivalently by the introduction of polymer systems.

3 Polymer Systems, Activities, Exponentiation

Polymer systems for statistical mechanics were introduced by Gruber and Kunz [GrK71]. This section presents a polymer system for the use of a RG analysis for field-theoretic models. The definition of polymer activities presented here uses block spin RG transformations. The difference to other definitions of RG transformations is the introduction of the $\mathcal{A}$-operator. An advantage in using block spin RG is that new polymer activities are defined by finite-dimensional integrals. A further advantage is that gradients of fields can be represented by gradients of the $\mathcal{A}$-operator, $\partial(\mathcal{A}\psi) = (\partial\mathcal{A})\psi$. For estimations of such gradients one has to bound gradients of $\mathcal{A}$ and there is no need of Sobolev-inequalities. A disadvantage is that the $\mathcal{A}$-operator is non-local and has to be taken into consideration for a RG step of polymer activities. A definition, not using block spin RG, similar to the one presented here can be found in Brydges contribution to these proceedings and in Brydges and Yau [BY90] and Dimock and Hurd [DH91, DH92, DH93].

Let $\Lambda = (a\mathbf{Z})^d$ be the d-dimensional hypercubic lattice with lattice spacing a. For a subset P of Λ denote by $|P|$ the number of elements in P. The union

$P \cup Q$ is denoted by $P + Q$ if P and Q are disjoint sets. The set consisting of all elements which are in P but not in Q is denoted by $P - Q$. Let $Pol(\Lambda)$ be a subset of the set of all finite subsets of Λ, $\mathcal{P}_{fin}(\Lambda) := \{P \subseteq \Lambda \,|\, |P| < \infty\}$, such that $P, Q \in Pol(\Lambda)$ implies $P + Q, P - Q \in Pol(\Lambda)$. $Pol(\Lambda)$ is called a *set of polymers*. Let Λ' be a block lattice of Λ. Suppose we have also defined a set of polymers $Pol(\Lambda')$. For simplicity suppose here that $Pol(\Lambda) := \mathcal{P}_{fin}(\Lambda)$ and $Pol(\Lambda') := \mathcal{P}_{fin}(\Lambda')$, i.e. polymers are finite subsets. We want to truncate the fluctuation propagator Γ and the $\mathcal{A}$-operator on polymers. For a polymer P let χ_P be the characteristic function. For $P \in Pol(\Lambda)$ and $X \in Pol(\Lambda')$ define truncated operators

$$\Gamma_P := \chi_P \Gamma \chi_P, \qquad \mathcal{A}_X := \mathcal{A}\chi_X \ .$$

For a polymer $X \in Pol(\Lambda')$ let us define a polymer $\overline{X} \in Pol(\Lambda)$ by

$$\overline{X} := \{y \in \Lambda \,|\, \exists x \in X : y \underline{\subseteq} x\}$$

and for a polymer $Y \in Pol(\Lambda)$ a polymer $[Y] \in Pol(\Lambda')$ by

$$[Y] := \{x \in \Lambda' \,|\, \exists y \in Y : y \underline{\subseteq} x\} \ .$$

Consider the *set of effective polymer partition functions* :

$$\mathcal{Z}(\Lambda, \mathcal{F}) := \{Z : Pol(\Lambda) \times \mathcal{F} \to \mathbf{R} \,|\, Z(\emptyset|\Psi) = 1\} \ , \tag{3}$$

where $\mathcal{F}$ is the set of fields, e.g. $\mathcal{F} := Fun(\Lambda) := \{F : \Lambda \to \mathbf{R}\}$. The RG transformation (without rescaling) for effective polymer partition functions is defined by

$$Z'(X|\Psi) := \int d\mu_{\Gamma_{\overline{X}}}(\zeta) Z(\overline{X}|\zeta + \mathcal{A}_X \Psi) \ ,$$

where $Z \in \mathcal{Z}(\Lambda, Fun(\Lambda))$ and $Z' \in \mathcal{Z}(\Lambda', Fun(\Lambda'))$. The RG transformation with rescaling is a mapping $\mathcal{RG} : \mathcal{Z}(\Lambda, Fun(\Lambda)) \to \mathcal{Z}(\Lambda, Fun(\Lambda))$, $Z' = \mathcal{RG}(Z)$.

The RG transformation can be performed in 4 steps. The first step is the *integration step* defined by the mapping

$$\mu_\Gamma : \mathcal{Z}(\Lambda, Fun(\Lambda)) \to \mathcal{Z}(\Lambda, Fun(\Lambda)), \quad \mu_\Gamma(Z)(Y|\phi) := \int d\mu_{\Gamma_Y}(\zeta) Z(Y|\zeta + \phi) \ .$$

The second step is the *coarsening step* defined by the mapping

$$[\,] : \mathcal{Z}(\Lambda, Fun(\Lambda)) \to \mathcal{Z}(\Lambda', Fun(\Lambda)), \quad [Z](X|\phi) := Z(\overline{X}|\phi) \ .$$

The third step is the *localization step* defined by the mapping

$$\iota_\mathcal{A} : \mathcal{Z}(\Lambda', Fun(\Lambda)) \to \mathcal{Z}(\Lambda', Fun(\Lambda')), \quad \iota_\mathcal{A}(Z)(X|\Psi) := Z(X|\mathcal{A}_X \Psi) \ .$$

The fourth and last step is the *rescaling step* defined by the mapping

$$\mathcal{R}_L : \; \mathcal{Z}(\Lambda', Fun(\Lambda')) \to \mathcal{Z}(\Lambda, Fun(\Lambda)), \quad \mathcal{R}_L(Z)(Y|\phi) := Z(LY|L^{1-\frac{d}{2}}\phi(\frac{\cdot}{L})) \; .$$

The mapping $\mathcal{RG}$ is therefore a composition of the four above defined mappings,
$\mathcal{RG} := \mathcal{R}_L \circ \iota_{\mathcal{A}} \circ [\,] \circ \mu_\Gamma$

$$
\begin{aligned}
\mathcal{Z}(\Lambda, Fun(\Lambda)) \;&\overset{\mu_\Gamma}{\to}\; \mathcal{Z}(\Lambda, Fun(\Lambda)) \overset{[\,]}{\to} \mathcal{Z}(\Lambda', Fun(\Lambda)) \\
&\overset{\iota_{\mathcal{A}}}{\to}\; \mathcal{Z}(\Lambda', Fun(\Lambda')) \overset{\mathcal{R}_L}{\to} \mathcal{Z}(\Lambda, Fun(\Lambda)) \; .
\end{aligned}
$$

Polymer partition functions obey the following conditions :

(a) Locality :
$$\frac{\partial}{\partial \Psi(y)} Z(P|\Psi) = 0, \quad \forall y \notin P$$

(b) Euclidean lattice symmetry :
$$Z(RY|R\Psi) = Z(Y|\Psi),$$

   $\forall R \in$ group of lattice symmetry

(c) Invariance under external symmetry transformations
   $\Psi \to U\Psi$:
$$Z(Y|U\Psi) = Z(Y|\Psi)$$

(d) Approximation (thermodynamic limit):
$$\lim_{Y \nearrow \Lambda} Z(Y|\Psi) = Z(\Psi)$$

For a control of polymer partition functions it is better to change to polymer activities. Polymer activities depend, like the polymer partition functions, on polymers and fields. The set of polymer activities is equivalent to the set of polymer partition functions. This means that if we know the polymer activities, then the polymer partition functions are determined and vice versa. The polymer activities have the following important property (at least for weakly coupled models) which the polymer partition functions do not share. The value of a polymer activity for a given polymer and field is small if the polymer contains a large number of elements or is of large extension. The polymer activities and polymer partition functions are related by an exponential function EXP. To define this function we will introduce a product $\circ$. The space, where this product is defined, is the *set of all polymer functions* defined by

$$\mathcal{M}(\Lambda, \mathcal{F}) := \{F : \; Pol(\Lambda) \times \mathcal{F} \to \mathbf{R}\} \; ,$$

where $\mathcal{F}$ is a set of fields. For notational simplicity we omit below the field-dependence of the polymer function. Define the $\circ$-product, for two polymer functions $U, V \in \mathcal{M}$, by

$$(U \circ V)(Y) := \sum_{\substack{P_1, P_2 \in Pol(\Lambda): \\ P_1 + P_2 = Y}} U(P_1) V(P_2) \ . \tag{4}$$

The sum is over all partitions of Y into two disjoint subsets. Defining an addition $+$ on $\mathcal{M}$ in the canonical way, we see that $(\mathcal{M}, \circ, +)$ is an associative algebra with unit element $\mathbf{1}$, $\mathbf{1}(P) := \delta_{P,\emptyset}$. Call this algebra a *polymer algebra* . Define on $\mathcal{M}$ a $\cdot$-multiplication by

$$(U \cdot V)(Y) := U(Y) V(Y) \ . \tag{5}$$

The set of all polymer activities $\mathcal{A}$ is a subset of the set of polymer functions $\mathcal{M}$, defined by

$$\mathcal{A}(\Lambda, \mathcal{F}) := \{A \in \mathcal{M} | \, A(\emptyset | \Psi) = 0\} \ .$$

$\mathcal{Z}, \mathcal{A} \subset \mathcal{M}$ are subalgebras of $(\mathcal{M}, \circ, +)$.

We are now in the position to define the exponential mapping $EXP : \mathcal{A} \to \mathcal{Z}$, where $\mathcal{Z}$ is the set of all polymer partition functions defined by eq. (3). For a polymer activity $A \in \mathcal{A}$, define

$$EXP(A) := \mathbf{1} + \sum_{n: \, n \geq 1} \frac{1}{n!} \underbrace{A \circ \cdots \circ A}_{n-times} \ . \tag{6}$$

EXP is bijective with inverse mapping $LN := EXP^{-1}$. Furthermore, $EXP : (\mathcal{A}, +) \to (\mathcal{Z}, \circ)$ is a group-isomorphism, i. e.

$$EXP(A + B) = EXP(A) \circ EXP(B) \ .$$

Let $Y \in Pol(\Lambda) - \{\emptyset\}$ be a nonempty polymer. Define the *set of all partitions* of Y by

$$\Pi(Y) \quad := \quad \bigcup_{n: \, n \geq 1} \Pi_n(Y),$$

$$\Pi_n(Y) \quad := \quad \{\{P_1, \ldots, P_n\} | \, Y = P_1 + \cdots + P_n\} \ .$$

The definition (6) of the exponential-function EXP and the fact that $(\mathcal{M}, \circ, +)$ is an associative algebra imply the following explicit representations

$$EXP(A)(Y) \quad = \quad \sum_{\mathbf{P} \in \Pi(Y)} \prod_{P \in \mathbf{P}} A(P)$$

$$LN(Z)(Y) \quad = \quad \sum_{n: \, n \geq 1} (-1)^{n-1} (n-1)! \sum_{\mathbf{P} \in \Pi_n(Y)} \prod_{P \in \mathbf{P}} Z(P) \ .$$

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For the proof that EXP is a bijective mapping one uses

$$A(Y) = EXP(A)(Y) - \sum_{\substack{\mathbf{P} \in \Pi(Y) \\ P \in \mathbf{P}: \, |P| < |Y|}} \prod_{P \in \mathbf{P}} A(P)$$

and proceeds by induction in the number of elements of a polymer. The RG transformation $\mathcal{RG}$ for polymer partition function can be transformed to a RG transformation $\widetilde{\mathcal{RG}}$ for polymer activities

$$\widetilde{\mathcal{RG}} := EXP^{-1} \circ \mathcal{RG} \circ EXP : \mathcal{A} \to \mathcal{A} \ .$$

This shows that $\mathcal{RG}$ and $\widetilde{\mathcal{RG}}$ are equivalent RG transformations. The RG flow of partition functions and activities are related by the following commutative diagram

$$
\begin{array}{ccccccccc}
\mathcal{Z} & \stackrel{\mu_\Gamma}{\to} & \mathcal{Z} & \stackrel{[\,]}{\to} & \mathcal{Z} & \stackrel{\iota_\mathcal{A}}{\to} & \mathcal{Z} & \stackrel{\mathcal{R}_L}{\to} & \mathcal{Z} \\
EXP \uparrow & & EXP \uparrow & & EXP \uparrow & & EXP \uparrow & & EXP \uparrow \\
\mathcal{A} & \stackrel{\widetilde{\mu_\Gamma}}{\to} & \mathcal{A} & \stackrel{\widetilde{[\,]}}{\to} & \mathcal{A} & \stackrel{\widetilde{\iota_\mathcal{A}}}{\to} & \mathcal{A} & \stackrel{\widetilde{\mathcal{R}_L}}{\to} & \mathcal{A}
\end{array}
$$

For a control of the RG flow of the polymer activities one has to study the four mappings $\widetilde{\mu_\Gamma}$, $\widetilde{[\,]}$, $\widetilde{\iota_\mathcal{A}}$ and $\widetilde{\mathcal{R}_L}$ in more detail.

3.1 Integration Step

New polymer activities $A' = \widetilde{\mu_\Gamma}(A)$ can be explicitly represented by tree graph formulas (cp. [B84, P86]). Instead of presenting here tree graph formulas, we introduce a recursive procedure by definition of a product (cp. also [BY90]).

For a representation of the mapping $\widetilde{\mu_\Gamma}$ one introduces the set of parametrized polymer activities

$$\mathcal{A}(\Lambda, [0,1], \mathcal{F}) := \{U : Pol(\Lambda) \times [0,1] \times \mathcal{F} \to \mathbf{C}\}$$

and special parametrized polymer activities A'_s, $s \in [0,1]$, by

$$EXP(A'_s) = \mu_{s\Gamma}(EXP(A)) \ . \tag{7}$$

This implies

$$A'_0 = A, \qquad A'_1 = \widetilde{\mu_\Gamma}(A) \ .$$

Define on the set of parametrized polymer activities a $\times_\Gamma$-product. For two parametrized polymer activities $A'_s, B'_s \in \mathcal{A}(\Lambda, [0,1], \mathcal{F})$ define the product

$$(A' \times_\Gamma B')_t(Y|\Psi) :=$$

$$\frac{1}{2} \sum_{\substack{Y_1, Y_2: \\ Y_1 + Y_2 = Y}} \sum_{\substack{y_1 \in Y_1 \\ y_2 \in Y_2}} \int_0^t ds \int d\mu_{(t-s)\Gamma}(\phi)$$

$$\frac{\partial}{\partial \Psi(y_1)} A'_s(Y_1|\phi + \Psi) \Gamma(y_1, y_2) \frac{\partial}{\partial \Psi(y_2)} B'_s(Y_2|\phi + \Psi) \ . \tag{8}$$

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We want to compute the parametrized activity $A'_t(Y|\Psi)$ by recursion in the number of elements of Y. Use the following definitions

$$\mathbf{1}_n(Y) := \left\{ \begin{array}{lll} 1 & : & |Y| = n, \\ 0 & : & |Y| \neq n \end{array} \right. \qquad \mathbf{1}_{>n} := \sum_{k:\, k>n} \mathbf{1}_k \ .$$

Then, the recursive equation is

$$A'_t = \mathbf{1}_{>1} \cdot [(A' \times_\Gamma A')_t + A] + \mathbf{1}_1 \cdot \mu_{t\Gamma}(A) \ . \tag{9}$$

The $\cdot$-product is defined by eq. (5). The proof of eq. (9) can be done in the following way. Firstly, distinguish the parametrized polymer activities A'_t, $t \in [0,1]$ defined by eq. (9) and eq. (7). Then show that they are equal. For $t = 0$, eq. (9) obeys $A'_0 = A$. By differentiation of eq. (7) with respect to s, we derive a first order differential equation of A'_s. Differentiation of eq. (9) with respect to t implies that A'_t, defined by eq. (9), obeys the same differential equation. The initial values at $t = 0$ are the same. Therefore, the parametrized polymer activities defined by eq. (7) and (9) are the same. Thus, eq. (9) is proven for all $t \in [0,1]$.

In the definition of the $\times_\Gamma$-product eq. (8) the right hand side depends only on polymers which contain lesser elements than the polymer on the left hand side of eq. (8). This implies that applying the activities in eq. (9) to special polymers, the recursion can be solved by a finite number of steps.

3.2 Coarsening Step

For polymer activities A on the lattice Λ, the *coarsened polymer activities* $\widetilde{[A]}$ are defined on the block lattice Λ' by

$$EXP(\widetilde{[A]}) = [EXP(A)] \ .$$

For a recursive computation of the coarsened polymer activities $\widetilde{[A]}$ define the following mapping $E_A : \mathcal{A} \to \mathcal{A}$, $A \in \mathcal{A}$

$$\begin{aligned} E_A(B)(Y) \quad := \quad & A(Y) + \sum_{P:\, y \in P \subseteq Y} \sum_{n:\, n \geq 1} \frac{1}{n!} \\ & \sum_{\substack{P_1,\ldots,P_n:\, [P]\cap[P_a]\neq\emptyset \\ P+P_1+\cdots+P_n=Y \ [P_a]\cap[P_b]=\emptyset,\, a\neq b}} A(P) \prod_{i=1}^n B(P_i) \ . \end{aligned}$$

In the definition of the mapping E_A we have chosen an element $y \in Y$, for all polymers Y. This choice of y is arbitrary. Define the polymer activity $\mathcal{C}A \in \mathcal{A}(\Lambda,\mathcal{F})$ by the following recursive equation

$$\mathcal{C}A = E_A(\mathcal{C}A) \ . \tag{10}$$

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That this equation defines $\mathcal{C}A$ recursively can be shown in the following way. Apply the activities in eq. (10) to a polymer Y. Then, $\mathcal{C}A(Y)$ can be expressed by a sum of products of terms $\mathcal{C}A(P)$, where P are polymers with $|P| < |Y|$. Thus, if $\mathcal{C}A(P)$ is defined for all polymers P containing less than $N = |Y|$ elements, then $(\mathcal{C}A)(Y)$ is defined by eq. (10). The recursion starts with the monomer (=polymer with only one element) $\mathcal{C}A(\{y\}) := A(\{y\})$, $y \in \Lambda$. Then, the coarsened polymer activity $\widetilde{[A]}(X) \in \mathcal{A}(\Lambda', \mathcal{F})$ is

$$\widetilde{[A]}(X) = (\mathcal{C}A)(\overline{X}) \ .$$

This equation can be proven in the following way. Firstly, it can be shown that the coarsened polymer activity obeys

$$\widetilde{[A]}(X) = \sum_{\substack{\mathbf{P}\in\Pi(\overline{X}): \\ \gamma(\{[P]|P\in\mathbf{P}\}) \ connected}} \prod_{P\in\mathbf{P}} A(P) \ ,$$

where, for $P_1, \ldots, P_n$, the Venn-diagram $\gamma(P_1, \ldots, P_n)$ is defined in the following way. Each polymer P_a is represented by a vertex and draw a line $(P_a P_b)$, $a \neq b$ if P_a and P_b are not disjoint, $P_a \cap P_b \neq \emptyset$. Take an arbitrary element y of $\overline{X}$ and take the polymer $Y \in \mathbf{P}$ which contains the element y. Then, the Venn-diagram $\gamma(\{[P]|P \in \mathbf{P}\} - \{Y\})$ decomposes into connected components. This decomposition implies the recursive equation (10).

3.3 Localization Step

The localization mapping $\widetilde{\iota_{\mathcal{A}}} : \mathcal{A} \to \mathcal{A}$ obeys, for all polymers $X \in Pol(\Lambda')$,

$$\sum_{X=\sum Q} \prod_Q A(Q|\mathcal{A}_X\phi) = \sum_{X=\sum Q} \prod_Q \widetilde{\iota_{\mathcal{A}}}(A)(Q|\phi) \ .$$

Define a parametrized polymer activity $A'_s(Q|\phi, \psi)$, for all $Q \in Pol(\Lambda')$, and two fields ψ and ϕ, implicitly by

$$\sum_{X=\sum Q} \prod_Q A(Q|s(\mathcal{A}_X\phi - \psi) + \psi) = \sum_{X=\sum Q} \prod_Q A'_s(Q|\phi, \psi) \ .$$

The parametrized polymer activity A'_s obeys, for the special values $s = 0, 1$,

$$A'_0(Q|\phi, \psi) = A(Q|\psi), \qquad A'_1(Q|\phi, \psi) = \widetilde{\iota_{\mathcal{A}}}(A)(Q|\phi) \ .$$

For two parametrized polymer activities A'_s and B'_s define the $\times_{\mathcal{A}}$-product by

$$(A' \times_{\mathcal{A}} B')_t(X|\phi, \psi) :=$$
$$\frac{1}{2} \sum_{\substack{X_1, X_2: \\ X_1+X_2=X}} \sum_{\substack{y\in\overline{X_1} \\ x\in X_2}} \int_0^t ds \, \mathcal{A}(y, x)\phi(x)\frac{\partial}{\partial\psi(y)}$$
$$A'_s\left(X_1|\phi, (t-s)(\mathcal{A}_X\phi - \psi) + \psi\right)$$
$$B'_s\left(X_2|\phi, (t-s)(\mathcal{A}_X\phi - \psi) + \psi\right) \ .$$

Then, the following recursive equation holds

$$A'_t = \mathbf{1}_{>1} \cdot [(A' \times_{\mathcal{A}} A')_t + A] + \mathbf{1}_1 \cdot \iota_{\mathcal{A},t}(A) \ .$$

We have seen how to control the RG steps given by the mappings $\widetilde{\mu_\Gamma}$, $\widetilde{[\ ]}$ and $\widetilde{\iota_{\mathcal{A}}}$ for the RG flow of polymer activities A by introducing products $\times_\Gamma$, $\times_{\mathcal{A}}$ and a recursive mapping E_A. A control of the RG flow of polymer activities A is achieved by splitting A into a relevant and an irrelevant part. This method of exact beta-function is discussed in the next section.

4   Exact Beta-Function Method

In general, a RG transformation $Z' = \mathcal{RG}(Z)$ maps a partition function Z, which is described by an infinite number of parameters to a partition function Z', which depends also on a infinite number of degrees of freedom. The exact beta-function method reduces the RG flow of an infinite number of parameters to a finite number of parameters called *coupling constants* (see [P93] for an applicication to hierarchical models). The remaining infinite number of parameters which determine the partition functions can be controlled by the coupling constants. The extraction of a finite number of coupling constants is done by a projection operator P such that

$$Z^{rel} = Z^{rel}(\gamma_0, \ldots, \gamma_N) = \mathrm{P}(Z)$$

depends on a finite number of parameters $\gamma_0, \ldots, \gamma_N$. The *irrelevant part* of the partition function Z is

$$R = (\mathbf{1} - \mathrm{P})(Z) \ .$$

The RG flow of this irrelevant part R can be controlled by standard fixed point theorems. The irrelevant fixed point $R^* = R^*(\gamma_0, \ldots, \gamma_N)$ depends on the coupling constants $\gamma_0, \ldots, \gamma_N$ and is defined by

$$R^* = H_{\gamma_0,\ldots,\gamma_N}(R^*) := (\mathbf{1} - \mathrm{P})\mathcal{RG}(Z^{rel}(\gamma_0, \ldots, \gamma_N) + R^*) \ .$$

Then, the exact beta-function $B : \mathbf{R}^{N+1} \to \mathbf{R}^{N+1}$ is defined by

$$Z^{rel}(B(\gamma_0, \ldots, \gamma_N)) = \\ \mathrm{P}\mathcal{RG}\left(Z^{rel}(\gamma_0, \ldots, \gamma_N) + R^*(\gamma_0, \ldots, \gamma_N)\right) \ .$$

The problem of searching fixed points $Z^* = \mathcal{RG}(Z^*)$ of the RG transformation is solved by finding fixed points $\gamma_0^*, \ldots, \gamma_N^*$ for the exact beta-function B, $B(\gamma_0^*, \ldots, \gamma_N^*) = (\gamma_0^*, \ldots, \gamma_N^*)$. Then, the fixed point Z^* is the sum of the relevant and irrelevant part at $(\gamma_0^*, \ldots, \gamma_N^*)$

$$Z^* = Z^{rel}(\gamma_0^*, \ldots, \gamma_N^*) + R^*(\gamma_0^*, \ldots, \gamma_N^*) \ .$$

For a control of the RG flow it is not necessary to determine the irrelevant fixed point R^* exactly. It is sufficient to find a neighborhood $U = U(R^*)$ of R^* which is stable under RG transformation, $H_{\gamma_0,\ldots,\gamma_N}(U) \subseteq U$. The beta-function method reduces the infinite-dimensional RG flow to a finite-dimensional RG flow of running coupling constants $\gamma = (\gamma_0, \ldots, \gamma_N)$:

$$\gamma \xrightarrow{B} \gamma' \xrightarrow{B} \gamma'' \xrightarrow{B} \cdots \ .$$

In the follwing subsections we will discuss the RG flow at the examples of the Φ^4-model and hierarchical fixed points.

4.1 Example: Φ_d^4-Model

Consider the RG transformations

$$V_{j-1}(\Psi) := -\ln \int d\mu_{v^j}(\Phi) \exp\{-V_j(\Phi + \Psi)\} - (\Psi = 0) \ ,$$

for all $j \in \{0, \ldots, n\}$, $n \in \mathbf{N}$. The effective interactions V_j are normalized by subtraction of a constant term, such that $V_j(0) = 0$. V_j is the effective interaction after $n - j$ RG steps. v^j is the fluctuation propagator for the $(n - j + 1)$th RG step. The starting (bare) interaction for the Φ^4-model in d dimensions is given by

$$V_n(\Phi) = \frac{1}{2}m_n^2 a_n^{-2} \int_z \Phi^2(z) + \frac{1}{2}\beta_n \int_z (\nabla\Phi)^2(z) + \frac{1}{4!}\lambda_n a_n^{d-4} \int_z \Phi^4(z) \ ,$$

where $a_n := L^{-n}a$, $a = $ unit length, $L \in \{2, 3, \ldots\}$. m_0^2, β_n, and λ_n are called bare coupling constants. Since the field Φ has dimension $a^{1-\frac{d}{2}}$, ∇ has dimension a^{-1} and $\int_z$ is of dimension a^d, we see that $\int_z \nabla^m \Phi^{2n}$ is of dimension $a^{2n-m-(n-1)d}$. Therefore, the constants m_n, β_n and λ_n are dimensionless. Running coupling constants m_j^2, β_j, λ_j and r_j, $j \in \{0, \ldots, n\}$ are defined by the following representation of the effective interactions

$$
\begin{aligned}
V_j(\Phi) &= \frac{1}{2}m_j^2 a_j^{-2} \int_z \Phi^2(z) + \frac{1}{2}\beta_j \int_z (\nabla\Phi)^2(z) \\
&+ \frac{1}{4!}\lambda_j a_j^{d-4} \int_z \Phi^4(z) + \frac{1}{6!}r_j a_j^{2d-6} \int_z \Phi^6(z) + \ldots \ .
\end{aligned}
$$

The RG flow of the running coupling constants m_j^2, β_j, λ_j and r_j is approximativly given by

$$
\begin{aligned}
m_{j-1}^2 &= L^2 m_j^2 + c_{21}\lambda_j - c_{22}\lambda_j^2 + O(\lambda_j^3) \\
\beta_{j-1} &= \beta_j - c_{2'2}\lambda_j^2 + O(\lambda_j^3) \\
\lambda_{j-1} &= L^{4-d}\lambda_j - c_{42}\lambda_j^2 + O(\lambda_j^3) \\
r_{j-1} &= L^{6-2d}r_j - c_{62}\lambda_j^2 + O(\lambda_j^3).
\end{aligned}
\tag{11}
$$

c_{mk}, $m \in \{2, 2', 4, 6\}$, $k \in \{1, 2\}$ are constants which depend on j and n (c_{21} depends only on j). For small λ_j, we see that m_j^2 is growing after RG steps. We call such a coupling constant *relevant*. The coupling constant β_j does not change (if we neglect $O(\lambda_j)$-terms) after RG steps. Such a coupling constant is called *marginal*. For $d > 3$ dimensions the coupling constant r_j becomes smaller after a RG step. Such a coupling constant is called *irrelevant*. We see by eq. (11) that the coupling constant λ_j is relevant, marginal and irrelevant for $d < 4$, $d = 4$ and $d > 4$ dimensions respectively.

Let us discuss the RG flow eqs. (11). Consider the case $2 \leq d \leq 4$. Let us choose

$$\lambda_n = L^{(d-4)n} \lambda$$

for the bare coupling constant. If we iterate the RG flow eq. (11) for m_j^2, we obtain $m_j^2 = m_j^2(\lambda_n, m_n^2)$. Each RG step produces a factor L^2 for m_j^2. Thus after $n - j$ RG steps we get a factor $L^{2(n-j)}$. To perform the ultraviolet limit $\lim_{n \to \infty}$ one has to dominate the factor L^{2n}. A term of order λ_n^k delivers a factor $L^{(d-4)kn}$ and therefore dominates L^{2n} if $(4-d)k > 2$. Thus terms of order λ^k for $k > \frac{2}{4-d}$ are not "dangerous" for the ultraviolet limit. For example, in $d = 2$ dimensions only the term $c_{21}\lambda_j$ on the right hand side of eq. (11) for m_j^2 produces divergent terms. To avoid such divergent terms on has to subtract this term from the starting coupling constant m_n^2. Such a subtraction term is called *counter term*. Generally, in $d < 4$ dimensions the counter terms can be expressed by a finite number of Feynman graphs. We call a model (ultraviolet) *super-renormalizable* if such a procedure is possible. The Φ_d^4-model is super-renormalizable in $d < 4$ dimensions. In $d = 4$ there are no suppression factors $L^{-\alpha}$, $\alpha > 0$, coming from terms containing powers of λ. Therefore, the counter terms cannot be expressed by a finite number of Feynman-graphs. Nevertheless, only a finite number of coupling constants are relevant. This property is called *strict renormalizability* of the Φ_4^4-model. For an existence proof of the infrared limit $\lim_{j \to -\infty}$ for the Φ_4^4-model one starts with coupling constants $m_0^2, \beta_0, \lambda_0$. The RG flow of the running coupling constant λ_j implies (for λ_0 small)

$$\lambda_j = O(\frac{1}{|j|}).$$

Therefore, $\lim_{j \to -\infty} \lambda_j = 0$ and the Φ_4^4-model becomes trivial in the infrared limit. Since $\sum_{j=0}^{-\infty} \lambda_j^k < \infty$ for $k > 1$, we see that terms of order λ^k, for $k > 1$, are "harmless" for the RG flow of running coupling constants.

4.2 Excursion: Hierarchical RG fixed points

A special application of the exact beta-function technique is the determination of hierarchical RG fixed points. The hierarchical RG transformations (HRGT) in d dimensions is given by a mapping $\mathcal{RG} : \{Z : \mathbf{R} \to \mathbf{R}\} \longrightarrow \{Z : \mathbf{R} \to \mathbf{R}\}$

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defined by

$$Z'(\Psi) := \mathcal{RG}(Z)(\Psi) := \left[\int d\mu_\gamma(\Phi) Z(\Phi + L^{1-\frac{d}{2}}\Psi) \right]^{L^d} ,$$

where the Gaussian measure $d\mu_\gamma(\Phi)$ is defined by, $\gamma > 0$,

$$d\mu_\gamma(\Phi) := (2\pi\gamma)^{-1/2} d\Phi \, e^{-\frac{\Phi^2}{2\gamma}} .$$

There exists fixed points Z^*, $Z^* = \mathcal{RG}(Z^*)$, such that the corresponding interactions $V^* := -\ln Z^*$ are l-wells, $l \in \{2, 3, \ldots\}$. It is well-known that the l-well fixed points exist in d dimensions, $2 \le d \le d_* = \frac{2l}{l-1}$ (cp. Collet and Eckmann (ϵ-expansion)[CE77, CE78]), Felder (RGDE) [F87]). A first rigorous construction of the 3-dimensional 2-well fixed point was accomplished by Koch and Wittwer [KW86, KW91].

For a representation of partition functions choose the following coordinates, supposing $\beta \ne 1$,

$$Z(\varphi) = \sum_{n=0}^{\infty} \frac{z_n}{\gamma'^n} : \varphi^{2n} :_{\gamma'} \leftrightarrow z = (z_0, z_1, \ldots) ,$$

where $\gamma' := \frac{\gamma}{1-\beta^2}$. The normal (Wick) ordering is defined by

$$: \varphi^{2n} :_\gamma := \exp\{-\frac{\gamma}{2}\frac{\partial^2}{\partial\varphi^2}\} \varphi^{2n} .$$

Define an associative $\times$-product on $\mathbf{R}^\infty$ by

$$(a \times b)_l = \sum_{m,n:\, |m-n| \le l \le m+n} \mathcal{C}_l^{mn} a_m b_n ,$$

where the structure coefficients $\mathcal{C}_l^{mn}$ obey

$$: \varphi^{2m} :_\gamma \cdot : \varphi^{2n} :_\gamma = \sum_{l:\, |m-n| \le l \le m+n} \gamma^{m+n-l} \mathcal{C}_l^{mn} : \varphi^{2l} :_\gamma .$$

Then, the HRGT can be represented as an L^d-fold product (cp. [PPW94, PW94])

$$\mathcal{RG}(Z) = \mathcal{S}_\beta(\underbrace{z \times \cdots \times z}_{L^d \ factors}) ,$$

where $\beta = L^{1-\frac{d}{2}}$ and $\mathcal{S}_\beta$ is defined by

$$(\mathcal{S}_\beta(z))_l := \beta^{2l} z_l .$$

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For the special case $L^d = 2$, $\beta = L^{1-\frac{d}{2}} = 2^{-\frac{d-2}{2d}}$ the fixed points z^* are solutions of the quadratic equation

$$z = z \times_\beta z := \mathcal{S}_\beta(z \times z) \ .$$

For $\beta \neq 1$ the product $\times_\beta$ is nonassociative.

Three solutions of the fixed point equation are immediately found. They are $0, 1$ and the high-temperature fixed point $z_{HT} = \mathcal{N} e^{-c_* \varphi^2}$.

There is a simple argument by ϵ-expansion to show that new non-trivial fixed points appear below $d_* = \frac{2l}{l-1}$ dimensions. Suppose that $z_* = \mathbf{1} + h$, where $\mathbf{1} := (1, 0, 0, \ldots)$, is a fixed point, i. e.

$$h = 2\mathbf{1} \times_\beta h + h \times_\beta h \ .$$

This is equivalent to

$$Uh = h \times h \tag{12}$$

where U is a diagonal matrix, $U = diag(1 - 2\beta^{2l})_{l=0,1,\ldots}$. By eq. (12) h can only be infinitesimal small if approximatively $h \in \ker U$. But the kernel of U is only non-trivial, $\ker U \neq \{0\}$, if $1 = 2\beta^{2l}$ or equivalently $d = \frac{2l}{l-1}$. In this case one sees that in lowest order $h_m = -\alpha \delta_{m,l}$, i. e. the fixed point partition function is

$$Z(\varphi) = 1 - \alpha : \varphi^{2l} :_\gamma + \ldots \ .$$

Below the critical dimensions $d = \frac{2l}{l-1}$ all terms $: \varphi^{2m} :_\gamma$ for $m \leq l$ become relevant.

Koch and Wittwer extracted the high-temperature fixed point out of the partition function. Then γ and β change to $L^{-2}\gamma$ and $L^{-2}\beta$ respectively. For their proof of the existence of the 2-well fixed point in 3 dimensions they used the norm, for the case $2\beta^2 < 1$,

$$\|z\|_\rho^{(1)} := \sum_{n=0}^{\infty} \sqrt{(2n)!} |a_n| \rho^n \ .$$

For ϵ-expansion one has to consider the case $2\beta^2 > 1$ and uses the norm (cp. Pordt and Wieczerkowski [PW94])

$$\|z\|_\rho^{(\infty)} := \sup_n (n! |a_n| \rho^n) \ .$$

In constructive field theory (cp. [BY90]) a large field regulator is used

$$\|z\|_\rho^{(c)} := \sup_{\phi \in \mathbf{R}} |e^{\rho \phi^2} Z(\phi)| \ .$$

These norms are algebra-norms for special values of ρ. This means that, for all a, b,

$$\|a \times_\beta b\|_\rho \leq \|a\|_\rho \cdot \|b\|_\rho \ .$$

The beta-function technique works for $N \geq N_0 = O(1) (= 7)$ using the projection operator $\mathrm{P}(z) := (z_0, \ldots, z_N, 0, 0, \ldots)$, for $z \in \mathbf{R}^\infty$.

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5 Relevant and Irrelevant Parts, Renormalization, Repolymerization

The choice of a suitable projection operator and the definition of running coupling constants is more complicated for the full model than for the hierarchical case. The choice of running coupling constants is model-dependent and we consider in this section only the case of the Φ^4-model. The way of choosing the running coupling constants is equivalent to an implementation of renormalization conditions. It can be shown that renormalization is only necessary for small polymers (cp. Brydges and Yau [BY90], Dimock and Hurd [DH91, DH92, DH93]).

In this section the questions of renormalization of polymer activities and repolymerization of the polymer system are discussed. Firstly, specify some subsets RP_n of the polymer set $Pol(\Lambda)$ for $n \in \{0, 2, 2', 4\}$. The sets RP_n are called *renormalization parts*. Only polymers which are contained in the renormalization parts are renormalized. The renormalization parts obey the following conditions

1. $Q \in RP_n \wedge P \subseteq Q \Rightarrow P \in RP_n$.

2. $\forall y \in \Lambda \, \exists U_y \in Pol(\Lambda) : \forall P \in RP_n, y \in P : P \subseteq U_y$.

3. $RP_0 \supseteq RP_2 \supseteq RP_{2'} \supseteq RP_4 \supseteq \cdots$.

4. RP_n preserves lattice symmetry.

For a polymer $Y \in Pol(\Lambda)$, choose a function $\delta V_Y(y|\Psi)$ such that $\delta V_Y = \mathbf{1}_1 \cdot \delta V_Y$. Then, define a polymer activity $R \in \mathcal{A}$ by

$$Z(Y|\Psi) = EXP(\delta V_Y + R)(Y|\Psi) \ . \tag{13}$$

We can choose δV_Y in such a way that R obeys the following renormalization conditions

$$R(Y_0|\Psi)|_{\Psi=0} = 0, \qquad \sum_{y' : y' \in Y} \frac{\partial^2}{\partial \Psi(y) \partial \Psi(y')} R(Y_2|\Psi)|_{\Psi=0} = 0, \dots \ , \tag{14}$$

for all $Y_n \in RP_n$, $y' \in Y_n$. R is called *irrelevant activity*. The functions δV_Y correspond to perturbative counter terms. We may compute the counter terms δV_Y such that R defined by eq. (13) obeys the renormalization conditions (14). For the field $\psi = 0$ and the renormalization part $Y \in RP_0$, we have

$$Z(Y|0) = EXP(\delta V_Y)(Y|0) = \prod_{y \in Y} \delta V_Y(y|0) \ .$$

This implies

$$\delta V_Y(y|0) = \exp\{ \sum_{P : y \in P \subseteq Y} \frac{\widetilde{\ln Z}(P|0)}{|P|} \} \ ,$$

where the Moebius transform $\widetilde{\ln Z}$ of $\ln Z$ is implicitly defined by

$$\ln Z(Y|\psi) = \sum_{P:\, P \subseteq Y} \widetilde{\ln Z}(P|\psi) \ , \tag{15}$$

for all $Y \in Pol(\Lambda)$. Eq. (15) defines $\widetilde{\ln Z}$ uniquely. We have $\widetilde{\ln Z}(\emptyset) = \ln Z(\emptyset)$. Suppose that $\widetilde{\ln Z}(P)$ is defined for all $P \in Pol(\Lambda)$ with $|P| < N$. We want to define $\widetilde{\ln Z}(Y)$ for $Y \in Pol(\Lambda)$ with $|Y| = N$. Eq. (15) implies

$$\widetilde{\ln Z}(Y) = \ln Z(Y) - \sum_{\substack{P:\, P \subset Y \\ |P| < N}} \widetilde{\ln Z}(P) \ . \tag{16}$$

Since the terms on the right hand side of eq. (16) are uniquely defined, $\widetilde{\ln Z}(Y)$ is uniquely defined.

Eqs. (13) and (14) imply, for all $Y \in RP_2 \subseteq RP_0$,

$$\frac{1}{2} \sum_{y' \in Y} \frac{\partial^2}{\partial \psi(y) \partial \psi(y')} Z(Y|\psi)|_{\psi=0} =$$

$$\frac{1}{2} \sum_{P:\, y \in P \subseteq Y} \sum_{y' \in P} \frac{\partial^2}{\partial \psi(y) \partial \psi(y')} A(P|\psi)|_{\psi=0} =$$

$$\frac{1}{2} \sum_{y' \in Y} \frac{\partial^2}{\partial \psi(y) \partial \psi(y')} \delta V_Y(y|\psi)|_{\psi=0} \ .$$

We have used here that $\frac{\partial}{\partial \psi(y)} A(P|\psi)|_{\psi=0} = 0$, which follows from the symmetry property $A(P|-\psi) = A(P|\psi)$. This implies, for the counter term,

$$\delta V_Y(y|\psi) \;\; = \;\; \delta V_Y(y|0) + \frac{1}{2} \sum_{P:\, y \in P \subseteq Y}$$

$$\sum_{y' \in P} \frac{\partial^2}{\partial \psi(y) \partial \psi(y')} A(P|\psi)|_{\psi=0} \, \psi^2(y) + \cdots \ .$$

We want to compute the irrelvant polymer activity R for a given counter term δV_Y. Eq. (13) implies

$$EXP(\delta V_Y) \circ EXP(R)(Y) = EXP(A)(Y) \ . \tag{17}$$

Let $\widetilde{\delta V}$ be the Moebius transform of δV

$$\delta V_Y = \sum_{P:\, P \subseteq Y} \widetilde{\delta V}_P \ . \tag{18}$$

Since $EXP(A + B) = EXP(A) \circ EXP(B)$ and $\circ$ is an associative product, eq. (17) implies

$$EXP(R)(Y) = EXP(-\delta V_Y) \circ EXP(A)(Y) \ . \tag{19}$$

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This implies the following explicit formula for the irrelevant polymer activity R

$$R(Y) \;=\; \sum_{P:\,P\subseteq Y} \sum_{\substack{y\in P\mapsto P_y\subseteq Y \\ y\in P_y}} \sum_{n\geq 0} \frac{1}{n!} \sum_{\substack{P_1,\ldots P_n\subseteq Y-P:\,(P_1+\cdots+P_n)\cup\bigcup_{y\in P} P_y=Y \\ \gamma(P_1,\ldots,P_n,P_y,y\in P)\ \ conn.}}$$

$$\prod_{y\in P} \widetilde{\delta V}_{P_y}(y) \prod_{a=1}^{n} A(P_a) \;. \tag{20}$$

The second sum on the right hand side of eq. (20) is over all polymers P_y, for all $y \in P$, such that $y \in P$. Since renormalization concerns only small polymers, we have for a polymer Y large enough, the relation $\delta V_Y = \delta V$, where

$$\delta V(y|\Psi) \;:=\; \sum_{P:\,y\in P} \widetilde{\delta V}_P(y|\Psi)$$

$$=\; c_0 + c_2 \Psi^2(y) + c_{2'}(\nabla\Psi)^2(y) + c_4 \Psi^4(y) \;.$$

$\delta V(y|\Psi)$ is called the *relevant part* of the interaction.

If one wants stability bounds or norms with large field regulators one has to replace the counter term $\delta V(y|\Psi)$ by $\exp\{\delta V(y|\Psi)\}$. The philosophy of renormalization and repolymerization stays the same as discussed in this section.

The advantage in using this repolymerization procedure is that the relevant part depends on a fewer number of parameters. Therefore, the beta-function is defined on a space of lower dimension than the space of partition functions. The counter terms δV_Y depend on all renormalization parts which are contained in the polymer Y. The relevant part δV depend on fewer terms. For the example of the Φ^4-model it is determined by the four running coupling constants $c_0, c_2, c_{2'}, c_4$.

To control the flow of the irrelevant activites R and the running coupling constants, we introduce a 5th RG step. This step is called *repolymerization step*. It changes the polymer system only for small polymers and therefore the thermodynamic limit is unchanged. The repolymerization step is the replacement of the counter terms δV_Y on the right hand side of eq. (13) by the relevant part δV

$$Z(Y|\Psi) \to Z^{rep}(Y|\Psi) := EXP(\delta V + R)(Y|\Psi) \;.$$

For $y \in \Lambda$ let U_y be the polymer in the 2nd condition for renormalization parts. For $P \supseteq U_y$, we have $Z(P|\Psi) = Z^{rep}(P|\Psi)$. The projection operator P for the beta-function method is defined by

$$\mathrm{P}(A) = \delta V, \qquad (\mathbf{1} - \mathrm{P})(A) = R \;.$$

6 Norm Estimations

For numerically calculations one has to truncate the infinite number of irrelevant polymer activities $R(P)$, $P \in Pol(\Lambda)$. Then, one has to estimate the

truncation error. The control of the RG flows can be done by using norm estimates. The most important problem for estimating polymer activities is the large field contributions or equivalently large factorials. If we would not allow large field contributions, then ordinary perturbation would converge. Consider, for example, a partition function for the 0-dimensional φ^4 field theory

$$Z(\psi) = \int d\varphi \, e^{-\frac{m^2}{2}\varphi^2 - \lambda(\varphi+\psi)^4} \quad , \tag{21}$$

where λ is a non-negative constant. If we integrate over all $\varphi \in \mathbf{R}$, we see that the integral will diverge for negative λ. Therefore, the series expansion in powers of λ has zero convergence radius. If we restrict the integration over φ to the finite interval $[-K, K]$, $K < \infty$, then the integral is also convergent for negative values of λ and the power series is convergent. The large field behaviour for the partition function Z is

$$Z(\psi) \sim e^{-c\psi^2}, \qquad \psi \to \infty \quad , \tag{22}$$

where $c = O(\lambda^{1/2})$. Thus

$$Z(\psi) = \sum_{n=0}^{\infty} z_n \psi^{2n}, \qquad z_n \sim n!^{-1}, \qquad n \to \infty \quad .$$

For $c, \kappa, \rho \in \mathbf{R}_+$, define the following norms

$$\|Z\|_{c,\kappa} := \sup_{\substack{\varphi,\psi\in\mathbf{R}: \\ |\psi|\leq\kappa}} \{e^{c\varphi^2} |Z(\varphi + i\psi)|\}$$

and

$$\|Z\|_\rho := \sum_{n=0}^{\infty} \sqrt{(2n)!}|z_n|\rho^n \quad .$$

For c and ρ small enough the norms of Z defined by eq. (21) are finite, $\|Z\|_{c,\kappa} < \infty$, $\|Z\|_\rho < \infty$. The Taylor coefficient z_n of ψ^{2n} for $Z_0(\psi) = e^{-\lambda\psi^4}$ is of order $1/\sqrt{n!}$. Therefore, the $\|\cdot\|_\rho$-norm of $Z_0(\psi)$ is not finite, $\|Z_0\|_\rho = \infty$. Thus, we are only allowed to consider norms of partition functions after at least one RG step !

Before a norm for polymer activities can be defined, some notations for multiindices have to be introduced. Call $m \in \mathcal{M} : \Lambda \to \mathbf{N}$ a *multiindex* if

$$|m| := \sum_{x\in\Lambda} m(x) < \infty \quad .$$

The *support* of the multiindex m is defined by $supp\, m := \{x \in \Lambda\,|\, m(x) \neq 0\}$. The factorial is defined by $m! := \prod_{y\in\Lambda} m(y)!$. Let $\varphi : \Lambda \to \mathbf{R}$ be a field and m a multiindex. Define a power by $\varphi^m := \prod_{y\in\Lambda} \varphi(y)^{m(y)}$. For two multiindices

$m, n \in \mathcal{M}$ define an order relation by $m \leq n$ iff $m(y) \leq n(y)$, for all $y \in \Lambda$. Let A be a local polymer activity and write

$$A(P|\Psi) = \sum_{\substack{m \in \mathcal{M}: \\ supp\, m \subseteq P}} a(P|m)\Psi^m \ .$$

For a polymer P let $T(P)$ be the set of all tree graphs with vertex set P. The tree bound of P is defined by

$$T_\mu(P) := \sup_{\tau \in T(P)} \exp\{-\mu a^{-1} \sum_{(yy') \in \tau} \|y - y'\|\} \ , \tag{23}$$

where $\mu > 0$. $\| \cdot \|$ is the euclidean norm in $\mathbf{R}^d$. For $0 < k_1, k_2 < 1$, $\mu, \rho > 0$ and define the norm of the polymer activity A

$$\|A\|_{k_1,k_2,\mu,\rho} := \sup_{y \in \Lambda} \left\{ \sum_{P:\, y \in P \in Pol(\Lambda)} \sum_{\substack{m \in \mathcal{M}: \\ supp\, m \subseteq P}} \frac{\sqrt{m!}|a(P|m)|}{k_1 k_2^{|P|-1} T_\mu(P)} \rho^{|m|} \right\} \ .$$

The definition of the norm contains two sums. A sum over all polymers P which contains an element y and a sum over all multiindices m whose support is contained in P. The terms in the sums contain a square root of the factorial m. This represents the large field behavior. The factor $k_1 < 1$ yields that terms containing a large number of polymer activities A are supressed. The factor $k_2 < 1$ yields that $A(P)$ is small for polymers P with a large number of elements. The factor $T_\mu(P)$ yields that $A(P)$ is small for polymers P with large extension, i.e. polymers which contain elements x, y such that $\|x - y\|$ is large.

The remainder of this section is technical. It concerns the question of how to estimate general polymer activities after integration, coarsening and localization step. By these methods one can control the RG flow of the irrelevant polymer activities R.

6.1   Integration Step

For a multiindex $m : \Lambda^2 \to \mathbf{N}$, let us introduce the notations

$$m_1(x) := \sum_y m(y, x), \qquad m_2(y) := \sum_x m(y, x) \ . \tag{24}$$

In the following we will use that

$$\int d\mu_\Gamma(\phi) P(\Phi + \Psi) = \exp\{\frac{1}{2} \sum_{x,y} \Gamma(x, y) \frac{\partial^2}{\partial \psi(x) \partial \psi(y)}\} P(\Psi) \ ,$$

where P is a polynom. Series expansion of the Gaussian measure yields

$$\exp\{\frac{1}{2}\sum_{x,y}\Gamma(x,y)\frac{\partial^2}{\partial\psi(x)\partial\psi(y)}\} = \sum_{m:\Lambda^2\to\mathbf{N}}\frac{1}{2^{|m|}}\frac{\Gamma^m}{m!}\partial_\psi^{m_1+m_2} \quad.$$

Define structure coefficients $C_\Gamma(m,n)$, for $m,n:\Lambda\to\mathbf{N}$ by

$$\int d\mu_\Gamma(\phi)(\phi+\psi)^n = \sum_{m:\,m\leq n}C_\Gamma(m,n)\psi^m \quad.$$

We have, for $n:\Lambda\to\mathbf{N}$,

$$
\begin{aligned}
\int d\mu_\Gamma(\phi)(\phi+\psi)^n &= \sum_{k:\Lambda^2\to\mathbf{N}}\frac{1}{2^{|k|}}\frac{\Gamma^k}{k!}\partial_\psi^{k_1+k_2}\psi^n \\
&= \sum_{k:\Lambda^2\to\mathbf{N}}\frac{1}{2^{|k|}}\frac{\Gamma^k}{k!}\frac{n!}{(n-k_1-k_2)!}\psi^{n-k_1-k_2} \\
&= \sum_m\sum_{\substack{k:\Lambda^2\to\mathbf{N}:\\ n=k_1+k_2+m}}\frac{1}{2^{|k|}}\frac{\Gamma^k}{k!}\frac{n!}{m!}\psi^m \quad.
\end{aligned}
$$

This implies

$$C_\Gamma(m,n) = \sum_{\substack{k:\Lambda^2\to\mathbf{N}:\\ n=k_1+k_2+m}}\frac{1}{2^{|k|}}\frac{\Gamma^k}{k!}\frac{n!}{m!} \quad.$$

Consider the polymer activity $A(P|\psi) = \sum_{\substack{n:\Lambda\to\mathbf{N}\\ supp\,n\subseteq P}}a(P|n)\psi^n$ and the Gaussian integral $A' = \mu_\Gamma(A)$. Suppose that a' are coefficients of the Taylor expansion of A'. Then

$$a'(P|m) = \sum_{n:\,m\leq n}C_\Gamma(m,n)a(P|n), \qquad a' = C_\Gamma a \quad.$$

The integration step is $EXP(A') = \mu_\Gamma(EXP(A))$. The effective polymer activity are computed by an integration step $A' = \widetilde{\mu_\Gamma}(A)$. Thus

$$a'(Y|m) = \sum_{Y=\sum P}\sum_{n:\,m\leq n}C_{\Gamma,\{P\}}(m,n)\prod_P a(P|n) \quad,$$

where

$$C_{\Gamma,\{P\}}(m,n) := \sum_{\substack{k:\Lambda^2\to\mathbf{N}:\,n=k_1+k_2+m\\ \gamma(supp\,k,\{P\})\,conn.}}\frac{1}{2^{|k|}}\frac{\Gamma^k}{k!}\frac{n!}{m!} \quad.$$

For estimations the trick of exponential pinning is important. Exponential pinning is given by the following bound. Consider $\alpha\in\mathbf{R}_+$, $b:\Lambda\to\mathbf{R}_+$, $m:\Lambda\to\mathbf{N}$. The multinominial theorem implies

$$b^m|m|!^\alpha \leq \|b^{1/\alpha}\|^{\alpha m}m!^\alpha, \qquad \|b\| := \sum_y b(y)$$

and, for $d : \Lambda^2 \to \mathbf{R}_+$, $m : \Lambda^2 \to \mathbf{N}$,

$$\frac{d^m}{m!} \leq \frac{\|d\|^{|m|}}{\sqrt{m_1! m_2!}} \quad,$$

where m_1 and m_2 are above defined by eq. (24). For the coefficients $a(P|n)$, $\rho \in \mathbf{R}_+$ and a polymer Y define the following norm

$$\|a(Y)\|_\rho := \sum_{m \in \mathcal{M}} \sqrt{m!}\, |a(Y|m)|\, \rho^m \quad.$$

For the mapping $b : \Lambda^2 \to \mathbf{R}_+$, define the norm $\|b\| := sup_y\{\sum_x b(y,x)\}$. By exponential pinning one can prove the following bound

$$\|a'(Y)\|_\rho \leq \sum_{Y = \sum P}\ \sup_{k:\, \gamma(supp\, k, \{P\})\ conn.} (\frac{|\Gamma|}{b})^k \prod_P \|a(P)\|_{2\sqrt{\frac{\|b\|}{2}}+\rho} \quad.$$

Using the definition eq. (23) of a tree bound, we obtain

$$T_\mu(P_1)\cdots T_\mu(P_n) \leq$$

$$\inf_{\tau \in T_n} \exp\left\{ \mu a^{-1} \sum_{(ab) \in \tau} dist(P_a, P_b) \right\} T_\mu(P_1 \cup \cdots \cup P_n) \quad.$$

Let $q > 0$, $b : \Lambda^2 \to \mathbf{R}_+$ and $c := \frac{|\Gamma|}{b} e^{\|\cdot - \cdot\|} : \Lambda^2 \to \mathbf{R}_+$ and suppose that

$$4\frac{k_1}{k_2}\|c\|\,\|A\|_{e^{-q}k_1, e^{-q}k_2, \mu, \rho'} < q^2 \quad,$$

where $\rho' := 2\sqrt{\frac{\|b\|}{2}} + \rho$. Then

$$\|\widetilde{\mu_\Gamma}(A)\|_{k_1, k_2, \mu, \rho} \leq \frac{\|A\|_{e^{-q}k_1, e^{-q}k_2, \mu, \rho'}}{1 - 4q^{-2}\frac{k_1}{k_2}\|c\|\|A\|_{e^{-q}k_1, e^{-q}k_2, \mu, \rho'}}$$

and

$$\|\mathbf{1}_{>1} \cdot (\widetilde{\mu_\Gamma}(A) - A)\|_{k_1, k_2, \mu, \rho} \leq$$

$$\frac{4q^{-2}\frac{k_1}{k_2}\|c\|\|A\|_{e^{-q}k_1, e^{-q}k_2, \mu, \rho'}}{1 - 4q^{-2}\frac{k_1}{k_2}\|c\|\|A\|_{e^{-q}k_1, e^{-q}k_2, \mu, \rho'}}\, \|A\|_{e^{-q}k_1, e^{-q}k_2, \mu, \rho'} \quad.$$

After an integration step the constants k_1 and k_2 become larger by a factor e^q. Thus, for a polymer P, we loose a factor $e^{q|P|}$. This factor can become very large if $|P|$ is large. In the next subsection, we will obtain a small factor for each element in the polymer after the coarsening step in the case where polymers are not small. The constant ρ grows to ρ'. For $d > 2$ dimensions this grow of ρ can be dominated by the scaling factor $L^{1 - \frac{d}{2}}$ after the rescaling step. For $d = 2$ one uses that the coupling constant λ (for Φ^4-model) grows by a factor L^2 after each RG step. This gives a supression factor $L^{-\frac{1}{2}}$ for the constant ρ which dominates the growing of ρ to ρ'.

6.2 Coarsening Step

Norm estimation of the coarsening step requires to distiguish between small and large polymers. We call a polymer X of the lattice Λ' with lattice spacing La *small* if, for all $y, x \in X$, the condition $\frac{|x^\mu - y^\mu|}{La} \in \{0, 1\}$, $\mu \in \{1, \ldots, d\}$, holds. Let $P_1, \ldots, P_n$ be polymers of Λ such that $X := [P_1 + \cdots + P_n]$ is not small and the Venn-diagram $\gamma([P_1], \ldots, [P_n])$ is connected. Then, there exists a positive $\epsilon = O(1)$ such that the following tree estimation holds

$$ k_1 \, k_2^{|P_i|-1} \prod_{i=1}^{n} T_\mu(P_i) \leq k_1 \, k_2^{|X|-1} T_\mu(X) \, (\frac{k_1}{k_2})^{n-1} \, (k_2 + e^{-\mu L})^{\epsilon |P_1 + \cdots + P_n|} \quad . \quad (25) $$

Let A be a polymer activity in Λ. The coarsened polymer activity $\widetilde{[A]}$ is given by

$$ \widetilde{[A]}(X) = \sum_{n:\, n \geq 1} \frac{1}{n!} \sum_{\substack{P_1, \ldots, P_n \in Pol(\Lambda): \\ P_1 + \cdots + P_n = \overline{X}, \ \gamma([P_1], \ldots, [P_n]) \ conn.}} \prod_{i=1}^{n} A(P_i) \quad . \quad (26) $$

For small k_2 and large L the factor $(k_2 + e^{-\mu L})^\epsilon$ is small. The tree estimation eq. (25) implies that for not small polymers $X \in Pol(\Lambda')$ the coarsened polymer activity $\widetilde{[A]}(X)$ is suppressed by $\gamma^{|X|}$ where γ is a small number. This small number can be used to supress the extra factor e^q for k_1 and k_2 after an integration step.

We estimate firstly the contributions to the norm of the coarsened polymer $\widetilde{[A]}$ coming from not small polymers. Then, we estimate the contributions coming from small polymers and the part of the right hand side of eq. (26) where $n \geq 2$. These two estimations will be discussed in this subsection. It remains to estimate contributions coming from small polymers and terms which contain at most one factor A. For these estimations one has to use the renormalization conditions to obtain supression factors. This method is well-known and not discussed here (see for example [R91]).

Define a projection operator P_{ns} defined on the set of all polymer activities on Λ which gives zero if applied to small polymers

$$ P_{ns}(A)(X) := \begin{cases} A(X) & : \quad X \text{ not small} \\ 0 & : \quad X \text{ small.} \end{cases} $$

For $d_1, \ldots, d_n \in \mathbf{N}$ such that $\sum_{a=1}^{n} d_a = 2(n-1)$ denote by $T_n(d_1, \ldots d_n)$ the tree graph with vertex set $\{1, \ldots, n\}$, d_i lines emerging from vertex i. Then Cayley's Theorem counts the number of tree graphs in $T_n(d_1, \ldots d_n)$

$$ |T_n(d_1, \ldots d_n)| = \frac{(n-2)!}{\prod_{i=1}^{n}(d_i - 1)!} \quad . \quad (27) $$

Cayley's Theorem and the estimation

$$\frac{|P_{ns}(\widetilde{[A]})(X)|}{k_1 k_2^{|X|-1} T_\mu(X)} \leq \sum_{n:\, n\geq 1} \frac{1}{(n-1)!} \sum_{\substack{d_1,\ldots,d_n:\\ \sum d_i = 2(n-1)}} \sum_{\tau \in T_n(d_1,\ldots d_n)}$$

$$\sum_{\substack{P_1,\ldots,P_n:\, x\in[P_1]\\ \gamma([P_1],\ldots,[P_n])\supseteq \tau,\ P_1+\cdots P_n=\overline{X}}} \prod_{i=1}^{n} \frac{A(P_i)}{k_1 k_2^{|X|-1} T_\mu(P_i)}$$

$$(\frac{k_1}{k_2})^{n-1} (k_2 + e^{-\mu L})^{\epsilon|P_1+\ldots+P_n|} \quad,$$

imply the following bound. Suppose that $4\frac{k_1}{k_2} L^d \|A\|_{\kappa^{-1}k_1,\kappa^{-1}k_2,\mu,\rho} < q^2$ holds, where $\kappa := e^{-q}(k_2 + e^{-\mu L})^\epsilon$. Then

$$\|P_{ns}(\widetilde{[A]})\|_{k_1,k_2,\mu,\rho} \leq \frac{\|A\|_{\kappa^{-1}k_1,\kappa^{-1}k_2,\mu,\rho}}{1 - 4q^{-2}\frac{k_1}{k_2} L^d \|A\|_{\kappa^{-1}k_1,\kappa^{-1}k_2,\mu,\rho}} \quad.$$

Denote by $\widetilde{[A]}_{>1}(X)$ the part of the right hand side of eq. (26) which consists of at least two A-factors and by $\widetilde{[A]}_1(X)$ the part which consists of only one A-term

$$\widetilde{[A]}_{>1}(X) = \sum_{n:\, n\geq 2} \frac{1}{n!} \sum_{\substack{P_1,\ldots,P_n \in Pol(\Lambda):\\ P_1+\cdots+P_n=\overline{X},\ \gamma([P_1],\ldots,[P_n])\ conn.}} \prod_{i=1}^{n} A(P_i)$$

$$\widetilde{[A]}_1(X) := \sum_{P:\, [P]=X} A(P) \quad.$$

Suppose that $4C\frac{k_1}{k_2} q^{-2} L^d \|A\|_{k_1,k_2,\mu,\rho} < 1$. Then, we have

$$\|(\mathbf{1} - P_{ns})(\widetilde{[A]}_{>1})\|_{k_1,k_2,\mu,\rho} \leq \frac{4Cq^{-2}\frac{k_1}{k_2} L^d \|A\|_{k_1,k_2,\mu,\rho}}{1 - 4Cq^{-2}\frac{k_1}{k_2} L^d \|A\|_{k_1,k_2,\mu,\rho}} \|A\|_{k_1,k_2,\mu,\rho} \quad.$$

6.3   *Localization Step*

Let $m : \Lambda \to \mathbf{N}$ be a multiindex and $\mathcal{A}$ be the operator defined in the block spin RG which maps fields defined on Λ to fields defined on Λ'. Expanding the mth power of $\mathcal{A}\phi$ yields

$$(\mathcal{A}\phi)^m = \sum_{n:\, \Lambda' \to \mathbf{N}} D(n,m)\phi^n \quad,$$

where

$$D(n,m) := \sum_{\substack{G:\, \Lambda\times\Lambda'\to \mathbf{N}\\ G_1=m,\ G_2=n}} \frac{m!}{G!} \mathcal{A}^G \quad.$$

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This is proven by

$$
\begin{aligned}
(\mathcal{A}\phi)^m &= \prod_{y\in\Lambda'} (\mathcal{A}\phi)(y)^{m(y)} = \prod_{y\in\Lambda'} \left(\sum_{x\in\Lambda'} \mathcal{A}(y,x)\phi(x) \right)^{m(y)} \\
&= \prod_{y\in\Lambda'} \left(\sum_{\substack{G_y:\Lambda'\to\mathbf{N} \\ |G_y|=m(y)}} \frac{m(y)!}{G_y} (\mathcal{A}(y,x)\phi(x))^{G_y(x)} \right) \\
&= \sum_{\substack{G:\Lambda\times\Lambda'\to\mathbf{N} \\ G_1=m}} \frac{m!}{G!} \mathcal{A}^G \phi^{G_2} \quad .
\end{aligned}
$$

For all multiindices $m:\Lambda\to\mathbf{N}$, let $a(m)$ be a real number. Define, for a multiindex $n:\Lambda'\to\mathbf{N}$

$$
a'(n) = \sum_{m:\Lambda\to\mathbf{N}} D(n,m)a(m) \quad .
$$

Then, we have

$$
\begin{aligned}
\|a'\|_\rho &\leq \sum_{n:\Lambda'\to\mathbf{N}} \sqrt{n!} \sum_{m:\Lambda\to\mathbf{N}} |D(n,m)|\,|a(m)|\rho^{|n|} \\
&\leq \sum_{m:\Lambda\to\mathbf{N}} \sum_{\substack{G:\Lambda\times\Lambda'\to\mathbf{N} \\ G_1=m,\ G_2=n}} \frac{|\mathcal{A}|^G}{G!} \sqrt{m!n!}\sqrt{m!}|a(m)|\rho^{|m|} \quad .
\end{aligned}
$$

Define, for $b:\Lambda\times\Lambda'\to\mathbf{R}_+$, the norm

$$
\|b\| := \max(\sup_{x\in\Lambda'} \sum_{y\in\Lambda} |b(y,x)|, \sup_{y\in\Lambda} \sum_{x\in\Lambda'} |b(y,x)|) < \infty \quad .
$$

Exponential pinning, for a multiindex $G:\Lambda\times\Lambda'\to\mathbf{N}$, yields

$$
\frac{b^G}{G!} \sqrt{m!n!} \leq \|b\|^{|G|} \quad ,
$$

where $G_1=m:\Lambda'\to\mathbf{N}$ and $G_2=n:\Lambda\to\mathbf{N}$ are defined by eq. (24). Thus

$$
\begin{aligned}
\|a'\|_\rho &\leq \sum_{m:\Lambda\to\mathbf{N}} \sum_{\substack{G:\Lambda\times\Lambda'\to\mathbf{N} \\ G_1=m,\ G_2=n}} (\frac{|\mathcal{A}|}{b})^G \|b\|^{|G|} \sqrt{m!}|a(m)|\rho^{|m|} \\
&\leq \sup_G (\frac{|\mathcal{A}|}{b})^G \|a\|_{\|b\|\rho} \quad .
\end{aligned}
$$

Let $A = LN(Z)$ be the polymer activity of Z. Define a polymer activity A' by

$$
Z(X|\mathcal{A}_X\phi) = \sum_{X=\sum Q} \prod_Q A'(Q|\mathcal{A}_Q\phi) \quad .
$$

Use the representations

$$A'(X|\psi) = \sum_{m:\,\Lambda'\to\mathbf{N}} a'(X|m)\psi^m$$

and

$$A(X|\phi) = \sum_{n:\,\Lambda\to\mathbf{N}} a(X|n)\phi^n \ .$$

Then, the following relation holds

$$a'(X|m) = \sum_{X=\sum Q} \sum_{n:\,\Lambda\to\mathbf{N}} D_{\mathcal{A},\{Q\}}(m,n) \prod_Q a(Q|n_Q) \ ,$$

where

$$D_{\mathcal{A},\{Q\}}(m,n) := \sum_{\substack{G:\,\Lambda\times\Lambda'\to\mathbf{N} \\ G_1=m,\,G_2=n,\,\gamma(supp\,G,\{Q\})\ conn.}} \frac{m!}{G!}\mathcal{A}^G \ .$$

The coefficients a and a' obey the following norm inequalities

$$\|a'(X)\|_\rho \le \sum_{X=\sum Q} \sup_{G:\,\gamma(supp\,G,\{Q\})\ conn.} \left(\frac{|\mathcal{A}|}{b}\right)^G \prod_Q \|a(Q)\|_{\|b\|_\rho} \ .$$

Define $c := \frac{|\mathcal{A}|}{b} e^{\mu\|\cdot-\cdot\|} : \Lambda\times\Lambda' \to \mathbf{R}_+$ and suppose, for $q > 0$,

$$4\frac{k_1}{k_2} q^{-2}\|c\|\|A\|_{e^{-q}k_1,e^{-q}k_2,\mu,\|b\|_\rho} < 1 \ .$$

Then

$$\|A'\|_{k_1,k_2,\mu,\rho} \le \frac{\|A\|_{e^{-q}k_1,e^{-q}k_2,\mu,\|b\|_\rho}}{1 - 4q^{-2}\frac{k_1}{k_2}\|c\|\|A\|_{e^{-q}k_1,e^{-q}k_2,\mu,\|b\|_\rho}} \ .$$

7 Conclusion

The input of a RG transformation starts with a given relevant part δV depending on a finite number of (running) coupling constants and an irrelevant polymer activity R. δV is a polymer function vanishing for polymers which contain more than 1 element. The polymer activity A is the sum of these two terms, $A = \delta V + R$. A new polymer activity $\widetilde{A}$ is defined after integration, coarsening, localization and rescaling step, $\widetilde{A} := \widetilde{\mathcal{R}_L} \circ \widetilde{\iota_{\mathcal{A}}} \circ \widetilde{[\,]} \circ \widetilde{\mu_\Gamma}(A)$. For a polymer X of Λ' the new irrelevant activity R' is defined by

$$EXP(\widetilde{A})(X) = EXP(\delta V_X + R')(X) \ ,$$

where the counter terms δV_X are defined by the polymer activities $\widetilde{A}$ such that R' fulfills renormalization conditions. The new (running) coupling constants

determine $\delta V'$. The sum of the new relevant part $\delta V'$ and the irrelevant polymer activity R' gives the new effective polymer activity A'.

The RG flow is splitted into a flow of the relevant part δV and irrelevant polymer activity R

$$\delta V \to \delta V'(\delta V, R), \qquad R \to R'(\delta V, R) \ .$$

The split of the RG flow is model-dependent. The methods for a definition and control of the irrelevant polymer activities is model-independent.

The control of the RG flow is solved by proving recursive bounds on the running coupling constants and the norms of the irrelevant polymer activities R. The definition of the norms has to contain parameters which control the size and extension of polymers and terms which control the large field behavior.

For an explicit control of the RG flow one proceeds as follows. We start with a bare interaction δV_n and an irrelevant polymer activity $R_n = 0$. For $j < n$ the effective interactions $\delta V_j^{(n)}$ and irrelevant activities $R_j^{(n)}$ are defined by the RG flows

$$\delta V_j^{(n)} \to \delta V_{j-1}^{(n)}(\delta V_j^{(n)}, R_j^{(n)}), \qquad R_j^{(n)} \to R_{j-1}^{(n)}(\delta V_j^{(n)}, R_j^{(n)}) \ .$$

$\delta V_j^{(n)}$ depends on certain coupling constants $\gamma_{0,j}^{(n)}, \dots, \gamma_{N,j}^{(n)} \in \mathbf{R}$. For the control of the RG flow one suposes that there exists finite intervals $I_{k,j}^{(n)} \subset \mathbf{R}$, for $k \in \{0, \dots, N\}$, $j \le n$, and constants $k_1, k_2, \mu, \rho, \epsilon_j^{(n)}$, for $k \in \{0, \dots, N\}$, $j \le n$ such that

$$\gamma_{k,j}^{(n)} \in I_{k,j}^{(n)}, \qquad \|R_j^{(n)}\|_{k_1, k_2, \mu, \rho} < \epsilon_j^{(n)}$$

implies

$$\gamma_{k,j-1}^{(n)} \in I_{k,j-1}^{(n)}, \qquad \|R_{j-1}^{(n)}\|_{k_1, k_2, \mu, \rho} < \epsilon_{j-1}^{(n)}.$$

For weakly coupled models, a guess of the intervals $I_{k,j}^{(n)}$ and constants $\epsilon_j^{(n)}$ can be achieved in the following way. Set $R_j^{(n)} = 0$ and compute the RG step

$$\delta V_j^{(n)} \to \delta V_{j-1}^{(n)}(\delta V_j^{(n)}, 0)$$

using perturbation theory. This gives an approximate relation of the flow of the coupling constants $\gamma_{0,j}^{(n)}$. For a guess of the constant $\epsilon_j^{(n)}$ use the RG step

$$R_j^{(n)} = 0 \to R_{j-1}^{(n)}(\delta V_j^{(n)}, R_j^{(n)} = 0)$$

and find a bound for $\|R_{j-1}^{(n)}(\delta V_j^{(n)}, R_j^{(n)} = 0)\|_{k_1, k_2, \mu, \rho}$.

Iterating the RG equations one sees that the relevant interaction terms $\delta V_j^{(n)}$ and the irrelevant activities $R_j^{(n)}$ depend on the starting (bare) interaction δV_n. For the infrared limit one has to show that δV_n can be defined such that the limits $\lim_{j \to -\infty} \delta V_j^{(n)}(\delta V_n)$ and $\lim_{j \to -\infty} R_j^{(n)}(\delta V_n)$ exist. For the ultraviolet limit one has to show that the limits $\lim_{n \to \infty} \delta V_j^{(n)}(\delta V_n)$ and $\lim_{n \to \infty} R_j^{(n)}(\delta V_n)$ exist.

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